Risk-Neutral versus Real Probabilities: A Little Brain-Teaser

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You are offered to take either side of the following bet:

- You win \$1,000,000 if the Dow Jones is lower than 10,000 in 6 months, otherwise you pay \$1,000,000

- If you take the opposite side of this bet, you win if the Dow Jones is higher than 10,000 in 6 months

Should you take such a bet or not? If yes, which side do you bet on? Before you answer this question, please read on . . .

Setup of the bet:

- The *real* Probability is

$$p^r \equiv \text{Prob}^{\text{real}}(\text{DJ} < 10,000) = 40\%$$

- The *risk-neutral* Probability is

$$p^n \equiv \text{Prob}^{\text{neutral}}(\text{DJ} < 10,000) = 51\%$$

- Suppose you are risk-averse

What is your decision? Is it riskfree? Do you need to make additional assumptions, e.g. as in the BSM model?
What if you are risk-neutral?

Two propositions:

- If you are *risk-averse*, you should care which bet is favored by the *risk-neutral* probabilities

- If you are *risk-neutral*, you should *usually* care about the *real-world* probabilities

For the first proposition we need to assume that either the conditions of the BSM model (in particular the possibility of continuous trading) hold *or* that we can freely trade in binary puts and calls on the Dow Jones with a strike of 10,000.

The economics: Why should the risk-neutral probabilities be relevant for a risk-averse player?

- The risk-neutral probabilities are driven by supply and demand for bets (derivatives)

- By following the risk-neutral probabilities the player does not take a stand in the bet. He just seizes an arbitrage opportunity between market prices (reflected by the risk-neutral probabilities) and the bet offered to him

In the BSM-world, there is a riskfree profit to be made:

We can evaluate the bet on the Dow being lower than 10,000 by calculating:

$$
\begin{aligned}
\text{Value of the bet} &= e^{-r_f T}\, E^n\{\text{Payoff}\} \\
&= e^{-r_f T}\, \{p^n\, 1,000,000 - (1 - p^n)\, 1,000,000\} \\
&\overset{!}{>} 0 \\
\Rightarrow p^n &\overset{!}{>} 50\%
\end{aligned}
$$

With $p^n = 51\%$ this bet has a strictly positive value, which can be locked in *at no risk* by taking the bet and hedging either dynamically or using binary options. In contrast, betting on DJ $> 10,000$ can be expected to win in 60% of all cases – but at the risk of loosing 40% of the time!

Some additional remarks:

- The bet on "DJ $< 10,000$" has the payoff of a portfolio which is long a binary put and short a binary call on the Dow Jones with strike 10,000. Note their valuation formulas, where p^n equals $1 - N(d_2)$ of the usual BSM option pricing formulas:
  $C_B = e^{-r_f T}(1 - p^n)$ and $P_B = e^{-r_f T}(p^n)$

- A risk-neutral better cares only about (real world) expectations. Of course, he could also lock in the profit of the riskfree strategy described above. If values of p^n and p^r are close, he would choose whichever bet is favored by either the real-world expectation *or* the risk-neutral expectation. (Can you work out the precise conditions?)

Epilog

- In their introduction "The parable of the bookmaker", Rennie and Baxter (1996, *Financial Calculus*, CUP) show how bookmakers quote their odds. Please note that our bet was quoted 1:1.

- Again the main point is that a bookmaker adjusts his odds by matching supply and demand for bets while disregarding (his subjective) views on the real probabilities. This way, his odds reflect the market view in the sense of an equilibrium of supply and demand – which is not to be confounded with a market wide information on what the the true probabilities were.